Introductory Lectures on Manifold Topology: Signposts

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Dedicated to my mother with deepest gratitude

– Y. S.
Preface

The purpose of this book is to introduce to advanced graduate students and other interested mathematicians some of the basic technique and results from manifold topology. It is assumed that the reader is familiar with algebraic topology through cup products and Poincaré duality as well as with fiber bundles and characteristic classes; e.g. with the material in the first half of the book “Characteristic Classes” by J. W. Milnor and J. D. Stasheff. A glance at the Contents shows the topics that are covered. The book is based on a course of lectures given by the first author during the fall semester, 2009 at the Morningside Center of the Chinese Academy of Sciences. It was originally planned as a year long course; hence some of the topics alluded to in the Introduction are not covered here. These will be done in a second volume.

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