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ALM 30: Automorphic Forms and L-functions
Automorphic Forms and $L$-functions

edited by

Jianya Liu
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Automorphic Forms and $L$-functions

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Preface

This is a collection of lecture notes from the CIMPA-UNESCO-CHINA Research School 2010: Automorphic Forms and $L$-functions, held at Shandong University at Weihai, in August 1–14, 2010.

The scientific directors were W. Duke and P. Sarnak (Chair) who provided detailed advice regarding the important aspects of the research school. The invited speakers were J. Cogdell, G. Harcos, E. Lapid, Xiaoqing Li, Wenzhi Luo, P. Michel, A. Reznikov, F. Shahidi, and Yangbo Ye.

Each of the speakers was scheduled to give a series of four lectures. Shahidi, although could not come to Weihai in person, has given full support to the research school by making his notes available, and Cogdell expertly combined his own lecture notes with Shahidi’s material to give eight lectures. Not only have the speakers given excellent lectures, but they have also spent time completing their lecture notes – available for us here. Besides the speakers, there were fifteen scholars from all over the world plus ten international graduate students and over sixty Chinese graduate students who participated in various activities offered by the research school. The diligence and enthusiasm, professional skills and hard work of speakers, participants and organizers have contributed to the great success of the research school.

M. Waldschmidt visited Weihai in 2007 and provided clear instructions on how to organize a CIMPA school. M. Jambu, representative of CIMPA, not only participated the whole school, but also conducted lots of administrative duties on behalf of CIMPA before and after the school.

It is a great pleasure to express my sincere thanks to all the friends who have made this research school possible and who have contributed to its success. Thanks are also due to Lizhen Ji, Liping Wang, and Huaying Li for their effort and patience, and to Taiyu Li for helping me prepare the TeX file for publication. This event would not have been possible without the financial support from Shandong University, the NSFC, and CIMPA-UNESCO, and I would like to express my gratitude and thanks to them.

Jianya Liu
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July 2013
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Wenzhi Luo, The Ohio State University
Philippe Michel, Ecole Polytechnique Fédérale de Lausanne
Andre Reznikov, Bar-Ilan University
Freydoon Shahidi, Purdue University
Yangbo Ye, The University of Iowa

School Coordinator
Jianya Liu, Shandong University
The CIMPA-UNESCO-CHINA Research School 2010: Automorphic Forms and $L$-functions
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**Lecture Notes on Some Analytic Properties of Automorphic L-functions for SL₂(Z)**

Yangbo Ye

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