Advanced Lectures in Mathematics
Volume 41

Handbook of Group Actions

Volume IV

Companion to the volumes
Handbook of Group Actions, Volume I
Handbook of Group Actions, Volume II
Handbook of Group Actions, Volume III

edited by

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ADVANCED LECTURES IN MATHEMATICS

Preface to Volumes III and IV

Although the mathematical literature is growing at an exponential rate, with papers and books published every day on various topics and with different objectives: research, expository or historical, the editors of the present Handbook feel that the mathematical community still needs good surveys, presenting clearly the bases and the open problems in the fundamental research fields. Group actions constitute one of these great classical and always active subjects which are beyond the fashionable and non-fashionable, at the heart of several domains, from geometry to dynamics, passing by complex analysis, number theory, and many others. The present Handbook is a collection of surveys concerned with this vast theory. Volumes I and II appeared in 2015.[1] Volume III and IV are published now simultaneously.

The list of topics discussed in these two volumes is broad enough to show the diversity of the situations in which group actions appear in a substantial way.

Volume III is concerned with hyperbolic group actions, groups acting on metric spaces of non-positive curvature, automorphism groups of geometric structures (complex, projective, algebraic and Lorentzian), and topological group actions, including the Hilbert–Smith conjecture and related conjectures.

Volume IV contains surveys on the asymptotic and large-scale geometry of metric spaces, presenting rigidity results in various contexts, with applications in geometric group theory, representation spaces and representation varieties, homogeneous spaces, symmetric spaces, and several aspects of dynamics: Property T, group actions on the circle, actions on Hilbert spaces and other symmetries.

Several surveys in these two volumes include new or updated versions of interesting open problems related to group actions.

We hope that this series will be a guide for mathematicians, from the graduate student to the experienced researcher, in this vast and ever-growing field.

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April 2018

[1] L. Ji, A. Papadopoulos and S. T. Yau (ed.). Handbook of Group actions, Volumes I and II, Higher Education Press and International Press, Vol. 31 and 32 of the Advanced Lectures in Mathematics, 2015.

Introduction to Volume IV

Volume 4 of the Handbook of Group Actions consists of three groups of chapters. The first group deals with the asymptotic (large-scale, or coarse) geometry of infinite groups and associated metric spaces, the second consists of homogeneous spaces, their quotient spaces, and discrete subgroups of Lie groups, and the third is concerned with group actions on nonlinear spaces, in particular ergodic theory and dynamics.

Part A: Asymptotic and Large-scale Geometry

One basic point of view in geometric group theory is to consider groups as metric spaces, and understand and relate algebraic properties of the groups to their metric properties. For a finitely generated group G, any finite symmetric generating set S defines a *proper* left invariant word metric d_S, and different generating sets define quasi-isometric metrics. On the other hand, if a group G is not finitely generated, any infinite generating set still defines a left invariant metric. But the metric is not proper anymore. The properness property is crucial for some applications. For any countable group H, by embedding it into a finitely generated group G, the restriction of a word metric of G to H defines a proper left invariant metric on H which is not a word metric. The chapter by Cornulier and de la Harpe gives an introduction to such proper left invariant metrics and more general pseudo-metrics on locally compact groups. The authors develop notions corresponding to those in the coarse geometry of finitely generated groups.

In complex analysis, conformal maps are basic, and they can be generalized to metric spaces using cross-ratios. The resulting maps are called Möbius maps. On the other hand, Möbius maps often do not exist between different spaces, and several generalizations have been proposed and studied. They include quasi-conformal maps, quasisymmetric maps and quasi-Möbius maps. The chapter by Haïssinsky gives a systematic study of actions of groups by quasi-Möbius maps, which are also called quasi-Möbius groups. This is a natural class of group actions for the following reasons: compared with actions by quasi-conformal maps, they are defined by global properties, while they have richer dynamics than actions by quasisymmetric maps. Consequently, they form a rich class of groups that sits between the more rigid class of conformal transformation groups and the too flexible class of group actions by homeomorphisms.

Though they are not conformal maps, quasi-Möbius maps capture the global behavior of conformal mappings on compact manifolds, and also capture the large scale geometry of groups acting by isometries on hyperbolic spaces.

After defining and discussing their basic properties, Haïssinsky addresses the following issues:

1. Determine when an action of a group by homeomorphisms is conjugate to that of a group by quasi-Möbius maps.
2. Determine when a group action by quasi-Möbius maps is conjugate to a group action by Möbius maps.
3. Describe the group of quasi-Möbius self-maps of a given metric space.
4. Classify metric spaces with large groups of quasi-Möbius maps.
5. Use quasi-Möbius group actions to describe the geometry of the spaces on which they act.

The notion of quasi-isometry is basic in geometric group theory. Two virtually isomorphic finitely generated groups are quasi-isometric with respect to any word metric. The question of quasi-isometric rigidity of a group Γ asks if any group which is quasi-isometric to it is virtually isomorphic to Γ. This question was actively studied in the 1990s for lattices of semisimple Lie groups. The chapter by Frigerio discusses the quasi-isometric rigidity of fundamental groups of manifolds that can be decomposed into pieces, in particular, it describes the behavior of the quasi-isometric rigidity of fundamental groups of three-manifolds with respect to the Milnor-Kneser prime decomposition and JSJ-decomposition. The quasi-isometric rigidity of fundamental groups of high-dimensional manifolds is also discussed.

One important source of geometric group theory is the Mostow strong rigidity for rank 1 locally symmetric spaces, especially hyperbolic manifolds, and the methods of its proof. In the chapter by Bourdon, a careful introduction to the Mostow strong rigidity of hyperbolic manifolds of finite volume and dimension at least 3 is given, together with a survey of related rigidity results and some recent results. The key idea is to consider actions of fundamental groups of hyperbolic manifolds on the boundary of hyperbolic space by quasi-conformal maps and to obtain regularity properties of the quasi-conformal actions. The methods can be applied to more general Gromov hyperbolic spaces and groups. Alternative proofs of the Mostow strong rigidity also lead to significant results, such as the entropy rigidity of Besson-Courtois-Gallot. This chapter gives a comprehensive survey of all these rigidity results.

Part B: Representation Spaces and Representation Varieties, Homogeneous Spaces, Symmetric Space, Mostow Rigidity

One important class of group actions consists of actions of discrete subgroups of Lie groups on homogeneous spaces, in particular symmetric spaces. The well developed Lie theory makes it possible to understand refined properties of these actions.

The classical example is the modular group $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper half plane which was already considered by Lagrange and Gauss. The group $\mathrm{SL}(2,\mathbb{Z})$ is an arithmetic subgroup of the Lie group $\mathrm{SL}(2,\mathbb{R})$, and a natural generalization is to consider arithmetic subgroups of semisimple Lie groups and their actions on symmetric spaces. More generally, we can consider lattices of semisimple Lie groups and their actions on symmetric spaces. These actions give rise to locally symmetric spaces of finite volume, which are important in the theory of automorphic forms and provide special Riemannian manifolds which often have rigidity properties as we saw in the previous chapter. The structure near infinity of these locally symmetric spaces and consequently these compactifications are well studied and understood. On the other hand, discrete subgroups which are not lattices occur right from the beginning of the theory of discrete subgroups, especially in the theory of Kleinian groups. They can be easily constructed from the uniformization theorem of Riemann surfaces. For a long time, it was not easy to construct discrete subgroups of semisimple Lie groups which are not lattices but are not contained in proper Lie subgroups. Recently, there was an explosion of activities around the notion of Anosov representations of hyperbolic subgroups and the resulting Anosov discrete subgroups, which provide a rich class of non-cofinite discrete subgroups of semisimple Lie groups, or equivalently locally symmetric spaces of infinite volume. One immediate problem is to understand compactifications of such locally symmetric spaces of infinite volume. The Chapter by Kapovich and Leeb gives a comprehensive introduction to these non-cofinite discrete subgroups and locally symmetric spaces of infinite volume.

Actions of linear algebraic groups on algebraic varieties arise naturally in several contexts, in particular, in the construction of moduli spaces by the method of geometric invariant theory, and to understand structures of toric varieties. One crucial step is to obtain local linearizations of such actions. More concretely, suppose an algebraic group G acts on a normal algebraic variety X by morphisms. Then the linearization of the action says that every point of X admits a G-stable neighborhood which is G-equivariantly isomorphic to an action of G on a G-stable subvariety of some projective space by linear transformations. This is an equivariant version of the basic fact that a projective variety is obtained by gluing affine subvarieties. A positive answer to the linearization problem was given by a theorem of Sumihiro. The chapter by Brion explains Sumihiro's theorem and related results.

In the geometric invariant theory as developed by Hilbert and Mumford, the key question is to construct quotients of actions of linear reductive algebraic groups on irreducible varieties. The chapter by Bérczi, Hawes, Kirwan and Doran describes the question on constructing geometric quotients of actions of general linear algebraic groups, i.e., not necessarily reductive, on irreducible varieties over algebraically closed fields of characteristic zero. This chapter is rather comprehensive and also introductory. Besides giving a survey of some recent work on geometric invariant theory and quotients of varieties by linear algebraic group actions, it also gives background material on linear algebraic groups. The linearization result discussed in the chapter by Brion plays a crucial role in this chapter.

Locally symmetric spaces form an important class of quotient spaces of infinite discrete group actions. Hyperbolic manifolds of dimension 3 of finite volume are special among locally symmetric spaces due to their rich geometry in connection with low dimensional topology. Reidemeister torsion was introduced to classify 3-dimensional lens spaces and also to give an important invariant of 3-dimensional manifolds. The chapter by Porti gives a systematic discussion of applications of Reidemeister torsion to hyperbolic manifolds of dimension 3. It also discusses Reidemeister torsion for representations of the fundamental group of hyperbolic manifolds of dimension 3 and hence view it as a function on the character variety of such fundamental groups. Group actions and representations of groups play a basic role in this chapter.

Homogeneous spaces and locally symmetric spaces often give rise to special manifolds, for example in the study of rigidity properties of manifolds. In the proofs of the Mostow strong rigidity and Margulis superrigidity, ergodic theory plays an important role. It turns out that the ergodic theory of group actions on homogeneous spaces is also important for the metric theory of Diophantine approximation. The chapter by Ghosh surveys some recent developments in the metric theory of Diophantine approximation with special emphasis on Diophantine properties of affine subspaces and their submanifolds.

The most basic group action is probably the regular action of a group on itself. A variant is the conjugation action. A conjugation is an automorphism of a group, and hence a natural extension is to consider the action on a group by a subgroup of its automorphism group. In the chapter by Dani, the author discusses actions on a connected Lie group by subgroups of its automorphism group. Such actions play an important role in the study of various topics, including geometry, dynamics, ergodic theory, probability theory on Lie groups, etc. After discussing basic results on the structure of the automorphism groups of a Lie group, ergodic theory and dynamics of such actions are discussed in this chapter.

In the study of rigidity of locally symmetric spaces and more general problems in ergodic theory, actions of locally compact groups on measure spaces occur naturally. They give rise to unitary representations of the groups on Hilbert spaces. The chapter by Bekka reviews results on the rigidity of these actions from the spectral point of view. More precisely, the rigidity means the existence of a spectral gap, i.e., the absence of almost invariant vectors, for associated averaging operators and their consequences. The spectral gap property has several striking applications to several topics, and this chapter explains two applications: construction of expanders graphs and bounds on the escape rate of random matrix products.

Part C: Dynamics: Property T, Group Actions on the Circle, Actions on Hilbert Spaces and Other Symmetries

The spectral gap property discussed in the previous chapter is defined and studied by actions of locally compact groups. The chapter by Valette deals with actions satisfying a property called a-(T)-menability. It is closely related to the Haagerup property, which is expressed in terms of unitary representations. These properties have applications to the Baum-Connes conjecture in the theory of C^*-algebras. The above concepts and applications are discussed in this chapter.

The most basic infinite group action is probably the action of $\mathbb{Z}$, and a natural extension is the action of $\mathbb{Z}^r$ for $r \geq 1$. It turns out that there is a marked difference in rigidity between the cases $r = 1$ and $r > 1$ when the action is hyperbolic in a suitable sense. Specifically, when $r > 1$, actions are rigid. This is consistent with the Mostow strong rigidity and the Margulis superrigidity for lattices in semisimple Lie groups of rank strictly greater than 1. The chapter by Hasselblatt explains such results.

Periodic functions such as $\sin x$ and $\cos x$ are essential for many applications. One generalization consists of almost periodic functions of one real variable, which roughly are periodic to within any desired level of accuracy with respect to suitably well-distributed "almost-periods". This notion has been generalized to locally compact abelian groups. The chapter by Veech deals with a generalization to discrete groups. It starts from the basic definitions and constructions such as the Bohr compactification, and then almost periodic functions are motivated by functions on the Bohr compactification. This chapter also discusses almost automorphic functions. One of the main results of this chapter relates almost automorphic functions to almost periodic functions. This chapter was written in 1970 and is also of historical value, besides giving an introduction to these basic notions.

The circle is probably the simplest compact connected manifolds, and in the study of group actions, actions on the circle are naturally believed to be among the most basic ones. It turns out that this case has many striking results and features. The chapter by Mann surveys rigidity results for groups acting on the circle. The main example concerns actions of surface groups and rigidity in both local and global settings. This chapter can serve as an introduction to recent developments and new tools in the study of group actions by homeomorphisms.

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